

# RANDOMNESS OF CHARACTER SUMS MODULO $m$

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**ABSTRACT.** Using a probabilistic model, based on random walks on the additive group  $\mathbb{Z}/m\mathbb{Z}$ , we prove that the values of certain real character sums are uniformly distributed in residue classes modulo  $m$ .

## 1. INTRODUCTION

A central question in number theory is to gain an understanding of character sums

$$S_\chi(x) = \sum_{n \leq x} \chi(n),$$

where  $\chi$  is a Dirichlet character modulo  $q$ . When  $q = p$  is a prime number and  $\chi_p = \left(\frac{\cdot}{p}\right)$  is the Legendre symbol modulo  $p$ , the character sums  $S_p(x) = S_{\chi_p}(x)$  encode information on the distribution of quadratic residues and non-residues modulo  $p$  (see for example Davenport and Erdős [5], and Peralta [13]). In particular, bounds for the order of magnitude of  $S_p(x)$  lead to results on the size of the least quadratic non-residue modulo  $p$  (see the work of Ankeny [2]; Banks, Garaev, Heath-Brown and Shparlinski [3]; Burgess [4]; Graham and Ringrose [6]; Lau and Wu [10]; Linnik [11]; and Montgomery [12]).

Quadratic residues and non-residues appear to occur in a rather random pattern modulo  $p$ , which suggests that the values of  $\chi_p(n)$  mimic a random variable that takes the values 1 and  $-1$  with equal probability  $1/2$ . This fact was recently exploited by Granville and Soundararajan [7] while investigating the distribution of the values of Dirichlet  $L$ -functions attached to quadratic characters at  $s = 1$ . Furthermore, a result of Davenport and Erdős [5] shows that short real character sums are indeed random in some sense. More specifically, they established that the values  $S_p(n + H) - S_p(n)$  are distributed according to a Gaussian distribution of mean zero and variance  $H$  as  $H \rightarrow \infty$  in the range  $\log H / \log p \rightarrow 0$  when  $p \rightarrow \infty$ .

In this paper, we investigate a new aspect of the *randomness* of these character sums. To describe our results, we first need some notation. Let  $F(X)$  be a square-free

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polynomial of degree  $d_F \geq 1$  over the finite field  $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ , and define

$$S_p(F, k) := \sum_{n \leq k} \chi_p(F(n)),$$

for all positive integers  $k \leq p$ . Moreover, let  $\Phi_p(F; m, a)$  be the proportion of positive integers  $k \leq p$  for which  $S_p(F, k) \equiv a \pmod{m}$ ; that is

$$\Phi_p(F; m, a) = \frac{1}{p} |\{k \leq p : S_p(F, k) \equiv a \pmod{m}\}|.$$

Since the values  $\chi_p(F(n))$  are expected to be randomly distributed, one might guess that  $\Phi_p(F; m, a) \sim 1/m$  for all  $a \pmod{m}$  as  $p \rightarrow \infty$ . We show that this is indeed the case in Corollary 1 below, uniformly for all  $m$  in the range  $m = o((\log p)^{1/4})$  as  $p \rightarrow \infty$ . Our strategy is to introduce a probabilistic model for the values  $S_p(F, k)$  based on random walks. A simple random walk on  $\mathbb{Z}$  is a stochastic process  $\{S_k\}_{k \geq 1}$  where

$$S_k = X_1 + \cdots + X_k,$$

and  $\{X_j\}_{j \geq 1}$  is a sequence of independent random variables taking the values 1 and  $-1$  with equal probability  $1/2$  (for further reference see Spitzer [14]). We shall model the values  $S_p(F, k) \pmod{m}$  by the stochastic process  $\{S_k \pmod{m}\}$  which may be regarded as a simple random walk on the additive group  $\mathbb{Z}/m\mathbb{Z}$ . To this end we consider the random variable

$$\Phi_{\text{rand}}(N; m, a) := \frac{1}{N} |\{k \leq N : S_k \equiv a \pmod{m}\}|.$$

Here and throughout  $\mathbb{E}(Y)$  will denote the expectation of the random variable  $Y$ . We first study the probabilistic model and prove

**Proposition 1.** *Let  $m \geq 2$  be a positive integer. Then, for all  $N \geq m^2$  we have*

$$\sum_{a=0}^{m-1} \mathbb{E} \left( \left( \Phi_{\text{rand}}(N; m, a) - \frac{1}{m} \right)^2 \right) \ll \frac{m^2}{N}.$$

Appealing to Markov's inequality, we deduce from this result that

$$\Phi_{\text{rand}}(N; m, a) = \frac{1}{m} (1 + o(1))$$

with probability  $1 - o(1)$  provided that  $N/m^2 \rightarrow \infty$ .

Using Proposition 1, we establish an analogous estimate for the second moment of the difference  $\Phi_p(F; m, a) - 1/m$  (which may be regarded as the “variance” of  $\Phi_p(F; m, a)$ ).

**Theorem 1.** *Let  $p$  be a large prime number and  $F(X) \in \mathbb{F}_p(X)$  be a square-free polynomial of degree  $d_F \geq 1$ . Then, for any integer  $2 \leq m \ll (\log p)^{1/4}$  we have*

$$\sum_{a=0}^{m-1} \left( \Phi_p(F; m, a) - \frac{1}{m} \right)^2 \ll_{d_F} \frac{m^2}{\log p}.$$

As a consequence, we obtain

**Corollary 1.** *Under the same assumptions of Theorem 1, we have uniformly for all  $0 \leq a \leq m-1$*

$$\Phi_p(F; m, a) = \frac{1}{m} + O_{d_F} \left( \frac{m}{\sqrt{\log p}} \right).$$

Let  $R_p(F, k)$  be the number of positive integers  $n \leq k$  such that  $F(n)$  is a quadratic residue modulo  $p$ , and similarly denote by  $N_p(F, k)$  the number of  $n \leq k$  for which  $F(n)$  is a quadratic non-residue mod  $p$ . Using a slight variation of our method we also prove that the values  $R_p(F, k)$  (and  $N_p(F, k)$ ) are uniformly distributed in residue classes modulo  $m$ . In this case, the corresponding probabilistic model involves random walks on the non-negative integers, where each step is 0 or 1 with equal probability. Define

$$\tilde{\Phi}_p(F; m, a) = \frac{1}{p} |\{k \leq p : R_p(F, k) \equiv a \pmod{m}\}|.$$

Then, using a similar result to Proposition 1 in this case (see Proposition 3.3 below) we establish

**Theorem 2.** *Let  $p$  be a large prime number and  $F(X) \in \mathbb{F}_p(X)$  be a square-free polynomial of degree  $d_F \geq 1$ . Then, for any integer  $2 \leq m \ll (\log p)^{1/4}$  we have*

$$\sum_{a=0}^{m-1} \left( \tilde{\Phi}_p(F; m, a) - \frac{1}{m} \right)^2 \ll_{d_F} \frac{m^2}{\log p}.$$

A similar result holds replacing  $R_p(F, k)$  with  $N_p(F, k)$ .

An important question in the theory of random walks on finite groups is to investigate how close is the distribution of the  $k$ -th step of the walk to the uniform distribution on the corresponding group (see for example Hildebrand [8]). In our case this corresponds to investigating the distribution of  $S_k \pmod{m}$ . Define

$$\Psi_{\text{rand}}(k; m, a) = \text{Prob}(S_k \equiv a \pmod{m}).$$

**Proposition 2.** *Let  $m \geq 3$  be an odd integer and  $0 \leq a \leq m-1$ . Then*

$$\Psi_{\text{rand}}(k; m, a) = \frac{1}{m} + O \left( \exp \left( -\frac{\pi^2 k}{3m^2} \right) \right).$$

This shows that the distribution of  $S_k$  is close to the uniform distribution on  $\mathbb{Z}/m\mathbb{Z}$  when  $m = o(k^{1/2})$  as  $k \rightarrow \infty$ . Although this result is classical (see for example Theorem 2 of Aldous and Diaconis [1]), we chose to include its proof for the sake of completeness.

We now describe an analogous result that we derive for character sums. Let  $N$  be large, and for each prime  $p \leq N$ , we consider the walk on  $\mathbb{Z}/m\mathbb{Z}$  whose  $i$ -th step corresponds to the value of  $\chi_p(q_i) \pmod{m}$ , where  $q_i$  is the  $i$ -th prime number. One might guess that as  $p$  varies over the primes below  $N$ , the distribution of the  $k$ -th step of this

walk will be close to the uniform distribution in  $\mathbb{Z}/m\mathbb{Z}$ , as  $N, k \rightarrow \infty$  if  $m = o(k^{1/2})$ . Define

$$S_k(p) = \sum_{j \leq k} \chi_p(q_j),$$

and

$$\Psi_N(k; m, a) = \frac{1}{\pi(N)} |\{p \leq N : S_k(p) \equiv a \pmod{m}\}|.$$

Here and throughout  $\log_j$  will denote the  $j$ -th iterated logarithm, so that  $\log_1 n = \log n$  and  $\log_j n = \log(\log_{j-1} n)$  for each  $j \geq 2$ . We prove

**Theorem 3.** *Fix  $A \geq 1$ . Let  $N$  be large, and  $k \leq A(\log_2 N)/(\log_3 N)$  be a positive integer. Then we have*

$$\Psi_N(k; m, a) = \Psi_{\text{rand}}(k; m, a) + O_A\left(\frac{1}{\log^A N}\right).$$

Hence, using Proposition 2 we deduce

**Corollary 2.** *Let  $m$  be an odd integer such that  $3 \leq m \leq k^{1/2}$ . Then under the same assumptions of Theorem 3 we have uniformly for all  $0 \leq a \leq m-1$  that*

$$\Psi_N(k; m, a) = \frac{1}{m} + O_A\left(\exp\left(-\frac{\pi^2 k}{3m^2}\right) + \frac{1}{\log^A N}\right).$$

We remark that under the assumption of the Generalized Riemann Hypothesis for Dirichlet  $L$ -functions, we can improve the range of validity of Theorem 3 to  $k \ll (\log N)/(\log_2 N)$ .

## 2. PRELIMINARY LEMMAS

In this section we collect together some preliminary results which will be useful in our subsequent work. Here and throughout we shall use the notation  $e_m(x) = \exp\left(\frac{2\pi i x}{m}\right)$ . Recall the orthogonal relation

$$(2.1) \quad \frac{1}{m} \sum_{t=0}^{m-1} e_m(tn) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{m}, \\ 0 & \text{otherwise.} \end{cases}$$

Our first lemma gives the classical bound for incomplete exponential sums over  $\mathbb{F}_p$  of the form

$$S_I(P_1, P_2) = \sum_{n \in I} \chi_p(P_1(n)) e_p(P_2(n)),$$

where  $I$  is a subinterval of  $\{0, 1, \dots, p-1\}$ , and  $P_1(X), P_2(X) \in \mathbb{F}_p[X]$ , such that  $P_1(X)$  is a nontrivial square-free polynomial.

**Lemma 2.1.** *Let  $p \geq 3$  be a prime number and  $I, P_1(X), P_2(X)$  be as above. Then we have*

$$|S_I(P_1, P_2)| \leq 2D\sqrt{p} \log p,$$

where

$$D = \deg P_1(X) + \deg P_2(X).$$

*Proof.* First if  $I = \{0, \dots, p-1\}$ , then  $S_I(P_1, P_2) = S(P_1, P_2)$  is a complete sum and the result follows from the classical Weil bound for exponential sums [15]:

$$(2.2) \quad |S(P_1, P_2)| \leq Dp^{1/2}.$$

Now, if  $I$  is proper subinterval of  $\{0, \dots, p-1\}$ , we shall use a standard procedure to express our incomplete sum in terms of complete sums of the same type. Using equation (2.1) we see that

$$S_I(P_1, P_2) = \sum_{n \bmod p} \chi_p(P_1(n)) e_p(P_2(n)) \left( \sum_{m \in I} \frac{1}{p} \sum_{t \bmod p} e_p(t(m-n)) \right).$$

Changing the order of summation and noting that the inner double sum is a product of two sums, one being a geometric progression and the other a complete exponential sum, we obtain

$$(2.3) \quad \begin{aligned} S_I(P_1, P_2) &= \frac{1}{p} \sum_{t \bmod p} \left( \sum_{m \in I} e_p(tm) \right) \left( \sum_{n \bmod p} \chi_p(P_1(n)) e_p(P_2(n) - tn) \right) \\ &= \frac{1}{p} \sum_{t \bmod p} F_I(t) S(P_1, \widetilde{P}_2), \end{aligned}$$

where  $\widetilde{P}_2(X) = P_2(X) - tX$  and  $F_I(t) = \sum_{m \in I} e_p(tm)$ . If  $t \equiv 0 \bmod p$  then  $F_I(t) = |I|$ . Otherwise if  $I = \{M+1, \dots, M+N\}$ , say, then

$$F_I(t) = \frac{e_p(t(M+1)) - e_p(t(M+N+1))}{1 - e_p(t)}.$$

Here the numerator has absolute value at most 2, while the absolute value of the denominator is  $2|\sin(t\pi/p)|$ . Hence

$$|F_I(t)| \leq \left| \sin \left( \frac{t\pi}{p} \right) \right|^{-1} \leq \left( 2 \left\| \frac{t}{p} \right\| \right)^{-1},$$

where  $\|\cdot\|$  stands for the distance to the nearest integer. As a set of representatives modulo  $p$  we choose  $\{-\frac{p-1}{2}, \dots, \frac{p-1}{2}\}$ , so that for  $t \neq 0$  in this set we have

$$(2.4) \quad |F_I(t)| \leq \frac{p}{2|t|}.$$

Now, we insert (2.2) and (2.4) in (2.3) to obtain

$$|S_I(P_1, P_2)| \leq \frac{D}{p^{1/2}} \left( |I| + \sum_{1 \leq |t| \leq \frac{p-1}{2}} \frac{p}{2|t|} \right) \leq 2D\sqrt{p} \log p.$$

This completes the proof of the lemma.  $\square$

The following lemma will be later used to prove that the product of distinct shifts of a square-free polynomial cannot be a square in  $\mathbb{F}_p(X)$ .

**Lemma 2.2.** *Let  $r \geq 2$ , and  $z_1, \dots, z_r$ , be distinct elements of  $\mathbb{F}_p$ . Moreover, let  $\mathcal{M}$  be a nonempty finite subset of the algebraic closure  $\overline{\mathbb{F}_p}$  of  $\mathbb{F}_p$  with  $4|\mathcal{M}| < p^{\frac{1}{r}}$ . Then there exists a  $j \in \{1, \dots, r\}$  such that the translate  $\mathcal{M} + z_j$  is not contained in  $\cup_{i \neq j} (\mathcal{M} + z_i)$ .*

*Proof.* Suppose that  $(z_1, \dots, z_r, \mathcal{M})$  provides a counterexample to the statement of the lemma. Then clearly for any nonzero  $t \in \mathbb{F}_p$ ,  $(tz_1, \dots, tz_r, t\mathcal{M})$  is also a counterexample.

We now use Minkowski's theorem on lattice points in a symmetric convex body to find a nonzero integer  $t$  such that

$$\begin{cases} |t| & \leq p-1 \\ \left\| \frac{tz_1}{p} \right\| & \leq (p-1)^{-\frac{1}{r}} \\ & \vdots \\ \left\| \frac{tz_r}{p} \right\| & \leq (p-1)^{-\frac{1}{r}} \end{cases}$$

Another way to express this is that there are integers

$$(2.5) \quad \begin{cases} |y_j| & \leq p(p-1)^{-\frac{1}{r}} \\ y_j & \equiv tz_j \pmod{p} \end{cases}$$

for any  $j \in \{1, \dots, r\}$ . Thus  $(y_1, \dots, y_r, t\mathcal{M})$  provides a counterexample. Now let  $j_0$  be such that

$$|y_{j_0}| = \max_{1 \leq j \leq r} |y_j|.$$

Choose  $\alpha \in t\mathcal{M}$  and consider the set  $\tilde{\mathcal{M}} = t\mathcal{M} \cap (\alpha + \mathbb{F}_p)$ . Then  $(y_1, \dots, y_r, \tilde{\mathcal{M}})$  will also be a counterexample.

Note that  $\alpha + \mathbb{F}_p$  can be written as a union of  $|\mathcal{M}|$  intervals whose endpoints are in  $\tilde{\mathcal{M}}$ . Let  $\{\alpha + a, \alpha + a + 1, \dots, \alpha + b\}$  be the longest of these intervals. Then

$$|b - a| \geq \frac{p}{|\tilde{\mathcal{M}}|} \geq \frac{p}{|\mathcal{M}|}.$$

By this, (2.5) and the hypothesis  $4|\mathcal{M}| < p^{\frac{1}{r}}$  we deduce

$$|b - a| > 4p^{1-\frac{1}{r}} > 2|y_{j_0}|.$$

Now the point is that if  $y_{j_0} > 0$  then  $\alpha + a + y_{j_0}$  belongs to  $\tilde{\mathcal{M}} + y_{j_0}$  but does not belong to  $\cup_{i \neq j_0} (\tilde{\mathcal{M}} + y_i)$ , while if  $y_{j_0} < 0$  then  $\alpha + b + y_{j_0}$  belongs to  $\tilde{\mathcal{M}} + y_{j_0}$  but does not belong to  $\cup_{i \neq j_0} (\tilde{\mathcal{M}} + y_i)$ . This completes the proof of the lemma.  $\square$

Using this lemma, we prove the following result

**Lemma 2.3.** *Let  $F(X) \in \mathbb{F}_p(X)$  be a square-free polynomial of degree  $d_F \geq 1$ . Let  $b_1, \dots, b_L$  be distinct elements in  $\mathbb{F}_p$  such that  $L < (\log p)/\log(4d_F)$ . Then, for any*

$a \in \mathbb{F}_p$  the polynomial

$$H(X) = \prod_{j=1}^L F(aX + b_j),$$

is not a square in  $\mathbb{F}_p(X)$ .

*Proof.* Let  $\alpha_1, \dots, \alpha_s$  be the roots of  $F(X)$  in  $\overline{\mathbb{F}}_p$ . Since  $F(X)$  is square-free then the  $\alpha_j$  are distinct and  $s = d_F$ . Let  $\mathcal{M} = \{a^{-1}\alpha_1, \dots, a^{-1}\alpha_s\}$ , and write  $z_j = -a^{-1}b_j$  for all  $1 \leq j \leq L$ . Then note that  $\mathcal{M} + z_j$  is the set of the roots of  $F(ax + b_j)$  in  $\overline{\mathbb{F}}_p$ . By our hypothesis it follows that  $4|\mathcal{M}| < p^{1/L}$ . Hence, we infer from Lemma 2.2 that there exists a  $j \in \{1, \dots, L\}$  such that at least one of the roots of  $F(ax + b_j)$  is distinct from all the roots of  $\prod_{l \neq j} F(ax + b_l)$ . This shows that  $H(X)$  is not a square in  $\mathbb{F}_p(X)$  as desired.  $\square$

### 3. RANDOM WALKS ON THE INTEGERS MODULO $m$

In this section we shall study the distribution of the random walk  $\{S_k \bmod m\}_{k \geq 1}$  and prove Propositions 1 and 2. To this end, we establish the following preliminary lemmas.

**Lemma 3.1.** *If  $m \geq 3$  is an odd integer, then*

$$(3.1) \quad \max_{1 \leq t \leq m-1} \left| \cos \left( \frac{2\pi t}{m} \right) \right| \leq 1 - \frac{\pi^2}{3m^2},$$

and

$$\max_{1 \leq t \leq m-1} |1 + e_m(t)| \leq 2 - \frac{\pi^2}{6m^2}.$$

*Proof.* We begin by proving the first assertion. If  $m \geq 5$  is odd, then

$$\max_{1 \leq t \leq m-1} \left| \cos \left( \frac{2\pi t}{m} \right) \right| = \cos \left( \frac{2\pi}{m} \right).$$

Moreover we know that  $\cos(x) \leq 1 - x^2/3$  for  $0 \leq x \leq \pi/2$ . This yields

$$\max_{1 \leq t \leq m-1} \left| \cos \left( \frac{2\pi t}{m} \right) \right| \leq 1 - \frac{4\pi^2}{3m^2}.$$

Now, when  $m = 3$  we have  $\max_{1 \leq t \leq 2} |\cos(2\pi t/m)| = \cos(\pi/m) \leq 1 - \pi^2/(3m^2)$ . This establishes the first part of the lemma.

Moreover, we have

$$|1 + e_m(t)|^2 = 2 + 2\cos(2\pi t/m) \leq 4 \left( 1 - \frac{\pi^2}{6m^2} \right),$$

which follows from (3.1). Therefore, using that  $\sqrt{1-x} \leq 1 - x/2$  for  $0 \leq x \leq 1$  we obtain the second assertion of the lemma.  $\square$

**Lemma 3.2.** *If  $m \geq 2$  is an integer, then*

$$\sum_{t=1}^{m-1} \sum_{1 \leq j_1 < j_2 \leq N} \cos\left(\frac{2\pi t}{m}\right)^{j_2-j_1} = O(m^3 N),$$

and

$$\sum_{t=1}^{m-1} \sum_{1 \leq j_1 < j_2 \leq N} \left(\frac{1 + e_m(t)}{2}\right)^{j_2-j_1} = O(m^3 N).$$

*Proof.* We prove only the first statement, since the proof of the second is similar. For  $d \in \{1, \dots, N-1\}$ , the number of pairs  $1 \leq j_1 < j_2 \leq N$  such that  $j_2 - j_1 = d$  equals  $N - d$ . Therefore, the sum we are seeking to bound equals

$$(3.2) \quad \sum_{t=1}^{m-1} \sum_{d=1}^{N-1} (N-d) \cos\left(\frac{2\pi t}{m}\right)^d.$$

First, when  $m$  is odd, Lemma 3.1 implies that the last sum is

$$\leq mN \sum_{d=1}^{N-1} \max_{1 \leq t \leq m-1} \left| \cos\left(\frac{2\pi t}{m}\right) \right|^d \leq \frac{mN}{1 - \max_{1 \leq t \leq m-1} \left| \cos\left(\frac{2\pi t}{m}\right) \right|} \leq \frac{3m^3 N}{\pi^2}.$$

Now, when  $m = 2r$  is even, then either  $\cos(\pi t/r) = -1$  or  $|\cos(\pi t/r)| < 1$ . In the latter case the proof of Lemma 3.1 implies that  $|\cos(\pi t/r)| \leq 1 - \pi^2/(3r^2)$ . Hence, in this case we obtain

$$\sum_{d=1}^{N-1} (N-d) \left| \cos\left(\frac{\pi t}{r}\right) \right|^d \ll m^2 N.$$

On the other hand if  $\cos(\pi t/r) = -1$ , then our sum become

$$\sum_{d=1}^{N-1} (N-d)(-1)^d \leq 2N.$$

This completes the proof. □

We begin by proving Proposition 2 first, since its proof is both short and simple.

*Proof of Proposition 2.* Recall that

$$\Psi_{\text{rand}}(k; m, a) = \text{Prob}(X_1 + \dots + X_k \equiv a \pmod{m}) = \frac{1}{2^k} \sum_{\substack{\mathbf{v}=(v_1, \dots, v_k) \in \{-1, 1\}^k \\ v_1 + \dots + v_k \equiv a \pmod{m}}} 1.$$

Hence, using (2.1) we deduce

$$(3.3) \quad \Psi_{\text{rand}}(k; m, a) = \frac{1}{2^k m} \sum_{\mathbf{v}=(v_1, \dots, v_k) \in \{-1, 1\}^k} \sum_{t=0}^{m-1} e_m\left(t(v_1 + \dots + v_k - a)\right).$$



The contribution of the term  $t = 0$  to the above sum equals  $1/m$ . Moreover, since  $\sum_{\alpha \in \{-1,1\}} e_m(\alpha t) = 2 \cos(2\pi t/m)$ , then the contribution of the remaining terms equals

$$\frac{1}{2^k m} \sum_{t=1}^{m-1} e_m(-at) \sum_{\mathbf{v}=(v_1, \dots, v_k) \in \{-1,1\}^k} e_m\left(t(v_1 + \dots + v_k)\right) = \frac{1}{m} \sum_{t=1}^{m-1} e_m(-at) \cos\left(\frac{2\pi t}{m}\right)^k.$$

Thus, the result follows upon using Lemma 3.1.  $\square$

*Proof of Proposition 1.* First, note that

$$\Phi_{\text{rand}}(N; m, a) = \frac{1}{N} \sum_{j=1}^N Y_j \quad \text{where} \quad Y_j = \begin{cases} 1 & \text{if } S_j \equiv a \pmod{m} \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, if  $\mathbf{v} = (v_1, \dots, v_N) \in \{-1, 1\}^N$ , then (2.1) yields

$$|\{1 \leq j \leq N : v_1 + \dots + v_j \equiv a \pmod{m}\}| = \frac{1}{m} \sum_{j=1}^N \sum_{t=0}^{m-1} e_m\left(t(v_1 + \dots + v_j - a)\right).$$

This implies

$$\begin{aligned} (3.4) \quad \mathbb{E} \left( \left( \Phi_{\text{rand}}(N; m, a) - \frac{1}{m} \right)^2 \right) &= \frac{1}{2^N} \sum_{\mathbf{v}=(v_1, \dots, v_N) \in \{-1,1\}^N} \left( \frac{1}{N} \sum_{\substack{1 \leq j \leq N \\ v_1 + \dots + v_j \equiv a \pmod{m}}} 1 - \frac{1}{m} \right)^2 \\ &= \frac{1}{2^N (mN)^2} \sum_{\mathbf{v}=(v_1, \dots, v_N) \in \{-1,1\}^N} \left| \sum_{j=1}^N \sum_{t=0}^{m-1} e_m\left(t(v_1 + \dots + v_j - a)\right) - N \right|^2. \end{aligned}$$

Now, expanding the summand on the RHS of (3.4) we derive

$$\begin{aligned} \left| \sum_{j=1}^N \sum_{t=0}^{m-1} e_m\left(t(v_1 + \dots + v_j - a)\right) - N \right|^2 &= \left| \sum_{j=1}^N \sum_{t=1}^{m-1} e_m\left(t(v_1 + \dots + v_j - a)\right) \right|^2 \\ &= \sum_{1 \leq t_1, t_2 \leq m-1} e_m(a(t_2 - t_1)) \sum_{1 \leq j_1, j_2 \leq N} e_m\left(t_1(v_1 + \dots + v_{j_1}) - t_2(v_1 + \dots + v_{j_2})\right). \end{aligned}$$

Hence, we infer from (2.1) that

$$\begin{aligned} (3.5) \quad &\sum_{a=0}^{m-1} \left| \sum_{j=1}^N \sum_{t=0}^{m-1} e_m\left(t(v_1 + \dots + v_j - a)\right) - N \right|^2 \\ &= m \sum_{t=1}^{m-1} \sum_{1 \leq j_1, j_2 \leq N} e_m\left(t((v_1 + \dots + v_{j_1}) - (v_1 + \dots + v_{j_2}))\right) \\ &= m^2 N + m \sum_{t=1}^{m-1} \sum_{1 \leq j_1 < j_2 \leq N} \left( e_m\left(t(v_{j_1+1} + \dots + v_{j_2})\right) + e_m\left(-t(v_{j_1+1} + \dots + v_{j_2})\right) \right). \end{aligned}$$

Inserting this estimate into (3.4), and using that  $\sum_{\alpha \in \{-1, 1\}} e_m(\alpha t) = 2 \cos(2\pi t/m)$ , we obtain

$$\sum_{a=0}^{m-1} \mathbb{E} \left( \left( \Phi_{\text{rand}}(N; m, a) - \frac{1}{m} \right)^2 \right) = \frac{1}{N} + \frac{2}{mN^2} \sum_{t=1}^{m-1} \sum_{1 \leq j_1 < j_2 \leq N} \cos \left( \frac{2\pi t}{m} \right)^{j_2 - j_1}.$$

The result follows upon using Lemma 3.2 to bound the RHS of the last identity.  $\square$

In order to prove Theorem 2 we require an analogous result to Proposition 1 in the case of a random walk on the non-negative integers, where each step is 0 or 1 (rather than  $-1$  or  $1$ ). To this end, we take  $\{\tilde{X}_j\}_{j \geq 1}$  to be a sequence of independent random variables taking the values 0 and 1 with equal probability  $1/2$ , and define

$$\tilde{S}_k = \tilde{X}_1 + \cdots + \tilde{X}_k,$$

and

$$\tilde{\Phi}_{\text{rand}}(N; m, a) = \frac{1}{N} |\{1 \leq j \leq N : \tilde{S}_j \equiv a \pmod{m}\}|.$$

Using a similar approach to the proof of Proposition 1 we establish:

**Proposition 3.3.** *Let  $m \geq 2$  be a positive integer. Then, for all  $N \geq m^2$  we have*

$$\sum_{a=0}^{m-1} \mathbb{E} \left( \left( \tilde{\Phi}_{\text{rand}}(N; m, a) - \frac{1}{m} \right)^2 \right) \ll \frac{m^2}{N}.$$

*Proof.* We follow closely the proof of Proposition 1. First, a similar analysis used to derive (3.4) allows us to obtain

$$\begin{aligned} (3.6) \quad & \mathbb{E} \left( \left( \tilde{\Phi}_{\text{rand}}(N; m, a) - \frac{1}{m} \right)^2 \right) \\ &= \frac{1}{2^N (mN)^2} \sum_{\mathbf{v}=(v_1, \dots, v_N) \in \{0, 1\}^N} \left| \sum_{j=1}^N \sum_{t=0}^{m-1} e_m \left( t(v_1 + \cdots + v_j - a) \right) - N \right|^2. \end{aligned}$$

Hence, using the identity (3.5) in equation (3.6) we get

$$\begin{aligned} (3.7) \quad & \sum_{a=0}^{m-1} \mathbb{E} \left( \left( \tilde{\Phi}_{\text{rand}}(N; m, a) - \frac{1}{m} \right)^2 \right) \\ &= \frac{1}{N} + \frac{1}{mN^2} \sum_{t=1}^{m-1} \sum_{1 \leq j_1 < j_2 \leq N} \left( \left( \frac{1 + e_m(t)}{2} \right)^{j_2 - j_1} + \left( \frac{1 + e_m(-t)}{2} \right)^{j_2 - j_1} \right) \\ &= \frac{1}{N} + \frac{2}{mN^2} \sum_{t=1}^{m-1} \sum_{1 \leq j_1 < j_2 \leq N} \left( \frac{1 + e_m(t)}{2} \right)^{j_2 - j_1}, \end{aligned}$$

upon noting that

$$\sum_{t=1}^{m-1} \left( \frac{1 + e_m(t)}{2} \right)^d = \sum_{r=1}^{m-1} \left( \frac{1 + e_m(-r)}{2} \right)^d,$$

by making the simple change of variables  $r = m - t$ . Appealing to Lemma 3.2 completes the proof.  $\square$

#### 4. CHARACTER SUMS WITH POLYNOMIALS: PROOF OF THEOREMS 1 AND 2

We begin by proving the following key proposition which establishes the required link with random walks. Let  $p$  be a large prime number and  $F(X) \in \mathbb{F}_p(X)$  be a square-free polynomial of degree  $d_F \geq 1$  in  $\mathbb{F}_p(X)$ . Moreover, let  $L \leq (\log p)/\log(4d_F)$  be a positive integer, and put  $N = \lfloor p/L \rfloor - 1$ . Furthermore, for any  $\mathbf{v} = (v_1, \dots, v_L) \in \{-1, 1\}^L$  we define

$$(4.1) \quad D_{p,F}(\mathbf{v}, L) = \{0 \leq s \leq N : \chi_p(F(sL + j)) = v_j \text{ for all } 1 \leq j \leq L\}.$$

**Proposition 4.1.** *Let  $p$ ,  $L$ , and  $F(X)$  be as above. Then for any  $\mathbf{v} = (v_1, \dots, v_L) \in \{-1, 1\}^L$  we have*

$$|D_{p,F}(\mathbf{v}, L)| = \frac{p}{2^L L} \left( 1 + O_{d_F}(p^{-1/10}) \right).$$

*Proof.* Let  $S$  be the set of non-negative integers  $0 \leq s \leq N$  such that  $F(sL + j) \neq 0$  for all  $1 \leq j \leq L$ . Then  $|S| = N + O_{d_F}(1)$ . Moreover, note that for  $s \in S$  we have

$$(4.2) \quad \frac{1}{2^L} \prod_{j=1}^L (1 + v_j \chi_p(F(sL + j))) = \begin{cases} 1 & \text{if } s \in D_{p,F}(\mathbf{v}, L), \\ 0 & \text{otherwise.} \end{cases}$$

This yields

$$|D_{p,F}(\mathbf{v}, L)| = \frac{1}{2^L} \sum_{s=0}^N \prod_{j=1}^L (1 + v_j \chi_p(F(sL + j))) + O_{d_F}(1).$$

Expanding the product on the RHS of the previous estimate, we find that  $|D_{p,F}(\mathbf{v}, L)|$  equals

$$(4.3) \quad \begin{aligned} & \frac{1}{2^L} \sum_{s=0}^N \left( 1 + \sum_{l=1}^L \sum_{1 \leq i_1 < i_2 < \dots < i_l \leq L} v_{i_1} \cdots v_{i_l} \chi_p(F(sL + i_1) \cdots F(sL + i_l)) \right) + O_{d_F}(1). \\ &= \frac{N}{2^L} + \frac{1}{2^L} \sum_{l=1}^L \sum_{1 \leq i_1 < \dots < i_l \leq L} v_{i_1} \cdots v_{i_l} \sum_{s=0}^N \chi_p(F(sL + i_1) \cdots F(sL + i_l)) + O_{d_F}(1). \end{aligned}$$

Since  $F(X)$  is a square-free polynomial, then it follows from Lemma 2.3 that the polynomial  $H_{i_1, \dots, i_l}(X) = F(LX + i_1) \cdots F(LX + i_l)$  is not a square in  $\mathbb{F}_p(X)$ . Therefore, using Lemma 2.1 with  $P_1(X) = H_{i_1, \dots, i_l}(X)$ ,  $P_2(X) = 0$  and  $I = \{0, \dots, N\}$ , we obtain

$$\left| \sum_{s=0}^N \chi_p(F(sL + i_1) \cdots F(sL + i_l)) \right| \leq 2d_F L \sqrt{p} \log p.$$

Inserting this bound in (4.3) we get

$$(4.4) \quad |D_{p,F}(\mathbf{v}, L)| = \frac{p}{2^L L} + O_{d_F}(L\sqrt{p}\log p),$$

which completes the proof.  $\square$

*Proof of Theorem 1.* Recall that

$$\Phi_p(F; m, a) = \frac{1}{p} |\{1 \leq k \leq p : S_p(F, k) \equiv a \pmod{m}\}|.$$

Let  $L = \lceil (\log p) / (\log(4d_F)) \rceil$ , and put  $N = \lfloor p/L \rfloor - 1$ . Moreover, for any  $0 \leq s \leq N$ , we define

$$M_L(s; m, a) = |\{1 \leq l \leq L : S_p(F, sL + l) \equiv a \pmod{m}\}|.$$

Then, note that

$$(4.5) \quad \left| \Phi_p(F; m, a) - \frac{1}{m} \right| \leq \frac{1}{p} \sum_{s=0}^N \left| M_L(s; m, a) - \frac{L}{m} \right| + O\left(\frac{L}{p}\right).$$

To bound the sum on the RHS of (4.5), we use the Cauchy-Schwarz inequality which gives

$$\left( \sum_{s=0}^N \left| M_L(s; m, a) - \frac{L}{m} \right| \right)^2 \leq (N+1) \sum_{s=0}^N \left( M_L(s; m, a) - \frac{L}{m} \right)^2.$$

Hence, combining this estimate with (4.5), we deduce

$$(4.6) \quad \left( \Phi_p(F; m, a) - \frac{1}{m} \right)^2 \ll \frac{N}{p^2} \sum_{s=0}^N \left( M_L(s; m, a) - \frac{L}{m} \right)^2 + \frac{L^2}{p^2}.$$

On the other hand, since  $S_p(sL + l) = S_p(sL) + \sum_{j=1}^l \chi_p(F(sL + j))$ , then

$$(4.7) \quad \sum_{a=0}^{m-1} \left( M_L(s; m, a) - \frac{L}{m} \right)^2 = \sum_{b=0}^{m-1} \left( \Delta_L(s; m, b) - \frac{L}{m} \right)^2,$$

where

$$\Delta_L(s; m, b) = |\{1 \leq l \leq L : \sum_{j=1}^l \chi_p(F(sL + j)) \equiv b \pmod{m}\}|.$$

Therefore, upon combining (4.6) and (4.7) we obtain

$$(4.8) \quad \sum_{a=0}^{m-1} \left( \Phi_p(F; m, a) - \frac{1}{m} \right)^2 \ll \frac{N}{p^2} \sum_{a=0}^{m-1} \sum_{s=0}^N \left( \Delta_L(s; m, a) - \frac{L}{m} \right)^2 + \frac{mL^2}{p^2}.$$

Now we evaluate the inner sum on the RHS of the previous inequality. Using (2.1) we get

$$\begin{aligned}
 (4.9) \quad \sum_{s=0}^N \left( \Delta_L(s; m, a) - \frac{L}{m} \right)^2 &= \frac{1}{m^2} \sum_{s=0}^N \left| \sum_{l=1}^L \sum_{t=0}^{m-1} e_m \left( t \left( \sum_{1 \leq j \leq l} \chi_p(F(sL + j)) - a \right) \right) - L \right|^2 \\
 &= \frac{1}{m^2} \sum_{s=0}^N \left| \sum_{l=1}^L \sum_{t=1}^{m-1} e_m \left( t \left( \sum_{1 \leq j \leq l} \chi_p(F(sL + j)) - a \right) \right) \right|^2 \\
 &= \frac{1}{m^2} \sum_{\mathbf{v} \in \{-1, 1\}^L} \left| \sum_{l=1}^L \sum_{t=1}^{m-1} e_m \left( t(v_1 + \cdots + v_l - a) \right) \right|^2 D_{p,F}(\mathbf{v}, L).
 \end{aligned}$$

Hence, using Proposition 4.1 along with the identity (3.4) obtained in the random walk setting, we derive

$$\begin{aligned}
 &\sum_{s=0}^N \left( \Delta_L(s; m, a) - \frac{L}{m} \right)^2 \\
 &= \frac{p}{2^L m^2 L} \sum_{\mathbf{v} \in \{-1, 1\}^L} \left| \sum_{l=1}^L \sum_{t=1}^{m-1} e_m \left( t(v_1 + \cdots + v_l - a) \right) \right|^2 (1 + O_{d_F}(p^{-1/10})) \\
 &= pL \mathbb{E} \left( \left( \Phi_{\text{rand}}(L; m, a) - \frac{1}{m} \right)^2 \right) (1 + O_{d_F}(p^{-1/10})).
 \end{aligned}$$

Finally, combining this estimate with (4.8) we obtain

$$\begin{aligned}
 \sum_{a=0}^{m-1} \left( \Phi_p(F; m, a) - \frac{1}{m} \right)^2 &\ll_{d_F} \sum_{a=0}^{m-1} \mathbb{E} \left( \left( \Phi_{\text{rand}}(L; m, a) - \frac{1}{m} \right)^2 \right) + \frac{m(\log p)^2}{p^2} \\
 &\ll_{d_F} \frac{m^2}{\log p},
 \end{aligned}$$

which follows from Proposition 1. This completes the proof.  $\square$

*Proof of Theorem 2.* We only prove the result for  $R_p(F, k)$ , since the proof for  $N_p(F, k)$  is similar. Define

$$\delta_F(j) = \begin{cases} 1 & \text{if } \chi_p(F(j)) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then, note that

$$R_p(F, k) = \sum_{j=1}^k \delta_F(j).$$

We follow closely the proof of Theorem 1. Let  $L = \lceil (\log p) / \log(4d_F) \rceil$ , and  $N = \lfloor p/L \rfloor - 1$ .

For any  $0 \leq s \leq N$  we define

$$\tilde{\Delta}_L(s; m, b) = |\{1 \leq l \leq L : \sum_{j=1}^l \delta_F(sL + j) \equiv b \pmod{m}\}|.$$

Then, similarly to the estimate (4.8) we obtain

$$(4.10) \quad \sum_{a=0}^{m-1} \left( \tilde{\Phi}_p(F; m, a) - \frac{1}{m} \right)^2 \ll \frac{N}{p^2} \sum_{a=0}^{m-1} \sum_{s=0}^N \left( \tilde{\Delta}_L(s; m, a) - \frac{L}{m} \right)^2 + \frac{m(\log p)^2}{p^2}.$$

Moreover, an analogous approach which leads to the identity (4.9) also gives

$$\sum_{s=0}^N \left( \tilde{\Delta}_F(s; m, a) - \frac{L}{m} \right)^2 = \frac{1}{m^2} \sum_{\mathbf{v} \in \{0,1\}^L} \left| \sum_{l=1}^L \sum_{t=1}^{m-1} e_m \left( t(v_1 + \dots + v_l - a) \right) \right|^2 \sum_{\substack{0 \leq s \leq N \\ \delta_F(sL+j)=v_j \\ \text{for all } 1 \leq j \leq L}} 1.$$

Remark that if  $F$  does not vanish in the interval  $[sL+1, sL+L]$  then

$$\delta_F(sL+j) = \frac{1 + \chi_p(F(sL+j))}{2},$$

for all  $1 \leq j \leq L$ . Hence, writing  $\tilde{\mathbf{v}} = (\tilde{v}_1, \dots, \tilde{v}_L)$  with  $\tilde{v}_j = 2v_j - 1$ , we deduce

$$\sum_{\substack{0 \leq s \leq N \\ \delta_F(sL+j)=v_j \\ \text{for all } 1 \leq j \leq L}} 1 = |D_p(\tilde{\mathbf{v}}, L, F)| + O_{d_F}(1) = \frac{p}{2^L L} \left( 1 + O_{d_F}(p^{-1/10}) \right),$$

which follows from Proposition 4.1. Thus, appealing to the identity (3.6) obtained in the random walk setting, we derive

$$\sum_{s=0}^N \left( \tilde{\Delta}_F(s; m, a) - \frac{L}{m} \right)^2 = pL\mathbb{E} \left( \left( \tilde{\Phi}_{\text{rand}}(L; m, a) - \frac{1}{m} \right)^2 \right) (1 + O_{d_F}(p^{-1/10})).$$

Therefore, inserting this estimate in (4.10) and using Proposition 3.3 we obtain

$$\begin{aligned} \sum_{a=0}^{m-1} \left( \tilde{\Phi}_p(F; m, a) - \frac{1}{m} \right)^2 &\ll_{d_F} \sum_{a=0}^{m-1} \mathbb{E} \left( \left( \tilde{\Phi}_{\text{rand}}(L; m, a) - \frac{1}{m} \right)^2 \right) + \frac{m(\log p)^2}{p^2} \\ &\ll_{d_F} \frac{m^2}{\log p}, \end{aligned}$$

as desired.  $\square$

## 5. CHARACTER SUMS OF FIXED LENGTH: PROOF OF THEOREM 3

We shall derive Theorem 3 from the following proposition

**Proposition 5.1.** *Fix  $A \geq 1$ . Let  $N$  be large, and  $k \leq A(\log_2 N)/(\log_3 N)$ . Then for any  $\mathbf{v} = (v_1, \dots, v_k) \in \{-1, 1\}^k$  we have*

$$\frac{1}{\pi(N)} |\{p \leq N : \chi_p(q_j) = v_j \text{ for all } 1 \leq j \leq k\}| = \frac{1}{2^k} \left( 1 + O_A \left( \frac{1}{\log^A N} \right) \right).$$

*Proof.* If  $\log N \leq p \leq N$  then

$$\frac{1}{2^k} \prod_{j=1}^k (1 + v_j \chi_p(q_j)) = \begin{cases} 1 & \text{if } \chi_p(q_j) = v_j \text{ for all } 1 \leq j \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we deduce that the number of primes  $p \leq N$  such that  $\chi_p(q_j) = v_j$  for all  $1 \leq j \leq k$ , equals

$$\begin{aligned}
 &= \frac{1}{2^k} \sum_{p \leq N} \prod_{j=1}^k (1 + v_j \chi_p(q_j)) + O(\log N) \\
 (5.1) \quad &= \frac{1}{2^k} \sum_{p \leq N} \left( 1 + \sum_{l=1}^k \sum_{1 \leq i_1 < \dots < i_l \leq k} v_{i_1} \cdots v_{i_l} \chi_p(q_{i_1} \cdots q_{i_l}) \right) + O(\log N) \\
 &= \frac{\pi(N)}{2^k} + \frac{1}{2^k} \sum_{l=1}^k \sum_{1 \leq i_1 < \dots < i_l \leq k} v_{i_1} \cdots v_{i_l} \sum_{p \leq N} \left( \frac{q_{i_1} \cdots q_{i_l}}{p} \right) + O(\log N).
 \end{aligned}$$

For  $1 \leq i_1 < \dots < i_l \leq k$  we let  $Q_{i_1, \dots, i_l} = q_{i_1} \cdots q_{i_l}$ . Then it follows from the prime number theorem that  $Q_{i_1, \dots, i_l} \leq \prod_{j \leq k} q_j = e^{k \log k (1+o(1))} \leq (\log N)^{A+o(1)}$ . On the other hand, quadratic reciprocity implies that  $\left( \frac{Q_{i_1, \dots, i_l}}{p} \right)$  is a character of modulus  $Q_{i_1, \dots, i_l}$  or  $4Q_{i_1, \dots, i_l}$ . Therefore, appealing to the Siegel-Walfisz Theorem (see Corollary 5.29 of Iwaniec-Kowalski [9]), we deduce

$$\sum_{p \leq N} \left( \frac{Q_{i_1, \dots, i_l}}{p} \right) \ll_A (Q_{i_1, \dots, i_l})^{1/2} \frac{N}{\log^{2A} N}.$$

Inserting this estimate in (5.1) completes the proof.  $\square$

*Proof of Theorem 3.* Using (2.1) we obtain

$$\begin{aligned}
 \Psi_N(k; m, a) &= \frac{1}{\pi(N)} |\{p \leq N : S_k(p) \equiv a \pmod{m}\}| \\
 (5.2) \quad &= \frac{1}{m\pi(N)} \sum_{p \leq N} \sum_{t=0}^{m-1} e_m(t(S_k(p) - a)) \\
 &= \frac{1}{m\pi(N)} \sum_{t=0}^{m-1} \sum_{\mathbf{v} \in \{-1, 1\}^k} e_m\left(t(v_1 + \dots + v_k - a)\right) \sum_{\substack{p \leq N \\ \chi_p(q_j) = v_j \text{ for } 1 \leq j \leq k}} 1
 \end{aligned}$$

Thus, appealing to Proposition 5.1 along with the identity (3.3) obtained in the random walk setting we derive

$$\begin{aligned}
 \Psi_N(k; m, a) &= \frac{1}{2^k m} \sum_{t=0}^{m-1} \sum_{\mathbf{v} \in \{-1, 1\}^k} e_m\left(t(v_1 + \dots + v_k - a)\right) + O_A\left(\frac{1}{\log^A N}\right) \\
 &= \Psi_{\text{rand}}(k; m, a) + O_A\left(\frac{1}{\log^A N}\right),
 \end{aligned}$$

which completes the proof.  $\square$

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